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Effects of radially varying moduli on stress distribution of nonhomogeneous anisotropic cylindrical bodies

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Abstract

In this study, the stress distribution in a nonhomogeneous anisotropic cylindrical body is investigated. Using equilibrium equations, Hooke's law and strain–displacement relations, a system of equations is obtained in cylindrical coordinates in terms of stress potentials where elastic properties change in radial direction. Young's and shear moduli are expressed as power functions of r and Poisson's ratios are kept constant. Closed-form solutions for stress potentials and stress distribution are obtained for an axisymmetric, orthotropic cylinder. Results are checked with FE results. A pressurized thick walled cylinder example is studied in details. Stresses in radial, tangential and axial directions and Von Mises stresses are plotted for different powers of r.

Keywords: Nonhomogeneous; Anisotropic; Functionally graded materials; Thick walled cylinder

1. Introduction

Ceramic coatings are used to increase heat, wear and corrosion resistance, and strength of metals. But, due to distinct interfaces introduced, they have their own shortcomings: there is a discontinuity where ceramic bonds to metal which makes the region more vulnerable to failure. This problem is thought to be overcome by varying the material properties smoothly from metal to ceramic through coating. This continuous change in properties improves the thermal and mechanical behaviors of the system. Materials that have a continuous change of properties from one point to another are called functionally graded materials (FGMs).

In literature, there are various studies on thermal and mechanical analyses of functionally graded systems. A few works are on functionally graded cylindrical bodies. Horgan and Chan (1999a), studied

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pressurized nonhomogeneous isotropic hollow circular cylinder and disk with the Young's modulus varying along the radial direction. Stress distribution and vibrational response of a functionally graded rotating disks are studied by Horgan and Chan (1999b) and by Güven and Çelik (2001). Temperature and stress distribution in a hollow cylinder subjected to thermal shock are analyzed numerically by Awaji and Sivakumar (2001). Jabbari et al. (2002) obtained axisymmetric mechanical and thermal stresses for a thick hollow cylinder where temperature and pressure are applied at the inner and outer surfaces. In a later study by Jabbari et al. (2003), nonaxisymmetric case of the previous problem has been solved using nonaxisymmetric temperature distribution by expanding displacements and temperature distribution in Fourier series. Thermal stresses are also studied for functionally graded hollow cylinder by Liew et al. (2003) dividing the whole cylinder into discrete homogeneous sub-cylinders along the radial direction. Durodola and Attia (2000) have obtained displacement and stresses for nonhomogeneous orthotropic rotating disk using numerical integration and finite element method. In literature, there are only a few closed-form solutions in the field of nonhomogeneous anisotropic bodies. For example, Kim and Paulino (2002) obtained exact solutions for orthotropic, exponentially and linearly graded plates with fixed grip, tension and bending loading under generalized plane stress conditions where elastic properties change along the width of the plate. Alshits and Kirchner (2001) studied cylindrically anisotropic, radially nonhomogeneous materials under various boundary conditions. They obtained displacements and first order stress functions in series form assuming a plane strain case and using Stroh formalism for a general material nonhomogeneity.

An early study of the problem of a homogeneous anisotropic cylindrical body is by Lekhnitskii (1963) where he formulated the problem in terms of stress potentials. A similar problem was investigated by Ting (1996). He solved separately circular tube and bar subjected to a uniform pressure, shearing, torsion and extension in terms of elastic stiffnesses. In a continuing paper, Ting (1999) derived solutions in terms of elastic compliances similar to Lekhnitskii's. Chen et al. (2000) added uniform temperature change along radial direction to Ting's problem. Tarn (2001) solved functionally graded anisotropic cylinder under thermomechanical loading using state space formulation.

The aim of this study is to analyze the effect of continuous nonhomogeneity (FGM) on anisotropic cylindrical bodies. Lekhnitskii's (1963) stress formulation is used and closed-form solutions are compared to FE results. As an example, a pressurized thick walled cylinder is studied in detail.

2. Stresses in a nonhomogeneous anisotropic cylindrical body

2.1. Governing equations

The body considered here is bounded by a cylindrical surface, possesses nonhomogeneous cylindrical anisotropy and the axis of anisotropy, z, is parallel to the generator of the cylindrical surface, z'. The principal axes of inertia are x' and y'. Polar axis that is parallel to x' is x. The center of gravity, O' is located at $x_{O'}, y_{O'}$ with respect to the x, y, z coordinate system. F_z is axial force, $M_{x'}, M_{y'}$ are bending moments and T is torsion as shown in Fig. 1 (Lekhnitskii, 1963).

The assumptions are:

(1) the elastic properties a_{ij}, α_i and β_{ij} (where a_{ij} which are given in terms of engineering properties in Eq. (A.2b) are the elements of the compliance matrix in generalized Hooke's law—Eq. (A.1), $\alpha_i = a_{i3}/a_{33}$ and $\beta_{ij} = a_{ij} - a_{i3}a_{j3}/a_{33}$) are functions of r (radial direction). For simplicity, they will be denoted as a_{ij}, α_i and β_{ij} instead of $a_{ij}(r)$, $\alpha_i(r)$ and $\beta_{ij}(r)$, throughout following derivations;
(2) there are no body forces;
(3) stresses do not depend on z direction, the axis of anisotropy.

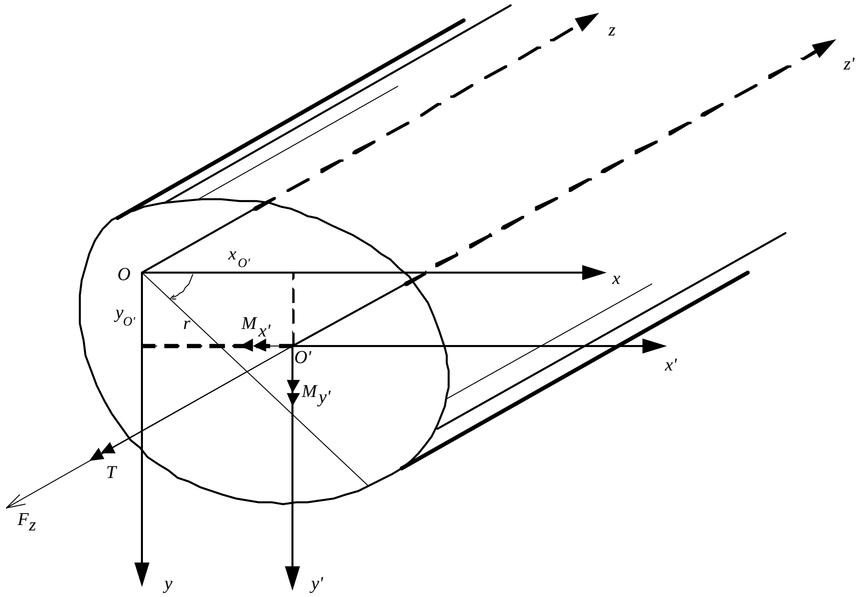


Fig. 1. Cylindrical body.

Integrating the z strains from strain–displacement relations in cylindrical coordinates with respect to z and using the generalized Hooke's law for strains except for ϵ_z (Lekhnitskii, 1963), displacements can be obtained as follows:

$$\left.\begin{aligned} u_r &= -\frac{z^2}{2}\frac{\partial\epsilon_z}{\partial r} + z\left(a_{15}\sigma_r + a_{25}\sigma_\theta + \cdots + a_{56}\tau_{r\theta} - \frac{\partial W}{\partial r}\right) + U, \\ u_\theta &= -\frac{z^2}{2}\frac{1}{r}\frac{\partial\epsilon_z}{\partial\theta} + z\left(a_{14}\sigma_r + a_{24}\sigma_\theta + \cdots + a_{46}\tau_{r\theta} - \frac{1}{r}\frac{\partial W}{\partial\theta}\right) + V, \\ w &= z\epsilon_z + W, \end{aligned}\right\} \tag{1}$$

where U, V and W are functions of r and θ.

Substituting the displacements above into the strain–displacement relations for ϵ_r, ϵ_θ and $\gamma_{r\theta}$, writing these strains in terms of stresses and equating the coefficients of z, z^2 to zero (because there should be no dependence on z) following equations are obtained from the coefficients of z^2:

$$\left.\begin{aligned} &\frac{\partial^2\epsilon_z}{\partial r^2} = 0, \\ &\frac{\partial\epsilon_z}{\partial r} + \frac{1}{r}\frac{\partial^2\epsilon_z}{\partial\theta^2} = 0, \\ &\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\epsilon_z}{r}\right) = 0, \end{aligned}\right\} \tag{2}$$

and from the coefficients of z:

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}\left(a_{15}\sigma_r + a_{25}\sigma_\theta + \cdots + a_{56}\tau_{r\theta} - \frac{\partial W}{\partial r}\right) = 0,\\
&\frac{\partial}{\partial \theta}\left(a_{14}\sigma_r + a_{24}\sigma_\theta + \cdots + a_{46}\tau_{r\theta} - \frac{1}{r}\frac{\partial W}{\partial \theta}\right)\\
&\quad + \left(a_{15}\sigma_r + a_{25}\sigma_\theta + \cdots + a_{56}\tau_{r\theta} - \frac{\partial W}{\partial r}\right) = 0,\\
&\frac{\partial}{\partial \theta}\left(a_{15}\sigma_r + a_{25}\sigma_\theta + \cdots + a_{56}\tau_{r\theta} - \frac{\partial W}{\partial r}\right)\\
&\quad + r\frac{\partial}{\partial r}\left(a_{14}\sigma_r + a_{24}\sigma_\theta + \cdots + a_{46}\tau_{r\theta} - \frac{1}{r}\frac{\partial W}{\partial \theta}\right)\\
&\quad - \left(a_{14}\sigma_r + a_{24}\sigma_\theta + \cdots + a_{46}\tau_{r\theta} - \frac{1}{r}\frac{\partial W}{\partial \theta}\right) = 0.
\end{aligned}\right\} \tag{3}$$

Using Eqs. (1)–(3), definitions of strains in terms of displacements and the generalized Hooke's law, one gets the following equations for ϵ_r, ϵ_θ and $\gamma_{r\theta}$:

$$\left.\begin{aligned}
&\epsilon_r = \frac{\partial U}{\partial r} = a_{11}\sigma_r + a_{12}\sigma_\theta + \cdots + a_{16}\tau_{r\theta},\\
&\epsilon_\theta = \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r} = a_{12}\sigma_r + a_{22}\sigma_\theta + \cdots + a_{26}\tau_{r\theta},\\
&\gamma_{r\theta} = \frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r} = a_{16}\sigma_r + a_{26}\sigma_\theta + \cdots + a_{66}\tau_{r\theta}.
\end{aligned}\right\} \tag{4}$$

Solving Eqs. (2), an expression for ϵ_z can be obtained as follows:

$$\epsilon_z = Ar\sin\theta + Br\cos\theta + C, \tag{5}$$

where A, B and C are arbitrary constants. Then, using the generalized Hooke's law and Eq. (5), σ_z is:

$$\sigma_z = \frac{1}{a_{33}}(Ar\sin\theta + Br\cos\theta + C) - \frac{1}{a_{33}}(a_{13}\sigma_r + a_{23}\sigma_\theta + a_{34}\tau_{\theta z} + a_{35}\tau_{rz} + a_{36}\tau_{r\theta}). \tag{6}$$

Substituting σ_z from the equation above into Eq. (3) and solving for $\frac{\partial W}{\partial r}$ and $\frac{1}{r}\frac{\partial W}{\partial \theta}$, one gets:

$$\left.\begin{aligned}
\frac{\partial W}{\partial r} &= \beta_{15}\sigma_r + \beta_{25}\sigma_\theta + \beta_{45}\tau_{\theta z} + \beta_{55}\tau_{rz} + \beta_{56}\tau_{r\theta}\\
&\quad + \alpha_5(Ar\sin\theta + Br\cos\theta + C) - D_1\sin\theta - D_2\cos\theta,\\
\frac{1}{r}\frac{\partial W}{\partial \theta} &= \beta_{14}\sigma_r + \beta_{24}\sigma_\theta + \beta_{44}\tau_{\theta z} + \beta_{45}\tau_{rz} + \beta_{46}\tau_{r\theta}\\
&\quad + \alpha_4(Ar\sin\theta + Br\cos\theta + C) - D_1\cos\theta + D_2\sin\theta - D_3 r,
\end{aligned}\right\} \tag{7}$$

similarly, from Eq. (4):

$$\left.\begin{aligned}
&\frac{\partial U}{\partial r} = \beta_{11}\sigma_r + \beta_{12}\sigma_\theta + \beta_{14}\tau_{\theta z} + \beta_{15}\tau_{rz} + \beta_{16}\tau_{r\theta} + \alpha_1(Ar\sin\theta + Br\cos\theta + C),\\
&\frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r} = \beta_{12}\sigma_r + \beta_{22}\sigma_\theta + \beta_{24}\tau_{\theta z} + \beta_{25}\tau_{rz} + \beta_{26}\tau_{r\theta} + \alpha_2(Ar\sin\theta + Br\cos\theta + C),\\
&\frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r} = \beta_{16}\sigma_r + \beta_{26}\sigma_\theta + \beta_{46}\tau_{\theta z} + \beta_{56}\tau_{rz} + \beta_{66}\tau_{r\theta} + \alpha_6(Ar\sin\theta + Br\cos\theta + C),
\end{aligned}\right\} \tag{8}$$

where

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}}, \quad \alpha_i = \frac{a_{i3}}{a_{33}} \qquad (i,j = 1,2,4,5,6) \tag{9}$$

and D_1, D_2, D_3 are arbitrary constants.

2.2. Stress function formulation

Stresses can be expressed in terms of stress potentials, $\widehat{F}(r,\theta)$ and $\hat{\psi}(r,\theta)$ (Lekhnitskii, 1963), which satisfy the equilibrium equations as follows:

$$\sigma_r = \frac{1}{r}\frac{\partial\widehat{F}(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\widehat{F}(r,\theta)}{\partial\theta^2}, \quad \sigma_\theta = \frac{\partial^2\widehat{F}(r,\theta)}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\widehat{F}(r,\theta)}{r}\right), \quad \tau_{rz} = \frac{1}{r}\frac{\partial\hat{\psi}(r,\theta)}{\partial\theta}, \quad \tau_{\theta z} = -\frac{\partial\hat{\psi}(r,\theta)}{\partial r}. \tag{10}$$

$\widehat{F}(r,\theta)$ and $\hat{\psi}(r,\theta)$ will be denoted as $\widehat{F}$ and $\hat{\psi}$ in the following equations for simplicity. Substituting stress potentials into Eqs. (7) and eliminating W by means of differentiation with respect to r and θ, one can obtain:

$$N_3''\widehat{F} + N_2'\hat{\psi} = \left[-\alpha_5 A + \left(2\alpha_4 + \frac{\mathrm{d}\alpha_4}{\mathrm{d}r}r\right)B\right]\cos\theta + \left[\left(2\alpha_4 + \frac{\mathrm{d}\alpha_4}{\mathrm{d}r}r\right)A + \alpha_5 B\right]\sin\theta + \left[\frac{\alpha_4}{r} + \frac{\mathrm{d}\alpha_4}{\mathrm{d}r}\right]C - 2D_3, \tag{11}$$

and similarly using Eqs. (8) and eliminating U and V, one gets:

$$\begin{aligned} N_4'\widehat{F} + N_3'\hat{\psi} = {} & [2(\alpha_1 - \alpha_2)A - 2\alpha_6 B]\frac{\sin\theta}{r} + \left[\left(\frac{\mathrm{d}\alpha_1}{\mathrm{d}r} - 4\frac{\mathrm{d}\alpha_2}{\mathrm{d}r}\right)A - \frac{\mathrm{d}\alpha_6}{\mathrm{d}r}B\right]\sin\theta \\ & + [2\alpha_6 A + 2(\alpha_1 - \alpha_2)B]\frac{\cos\theta}{r} + \left[\frac{\mathrm{d}\alpha_6}{\mathrm{d}r}A + \left(\frac{\mathrm{d}\alpha_1}{\mathrm{d}r} - 4\frac{\mathrm{d}\alpha_2}{\mathrm{d}r}\right)B\right]\cos\theta \\ & + \left[\frac{\mathrm{d}\alpha_1}{\mathrm{d}r} - 2\frac{\mathrm{d}\alpha_2}{\mathrm{d}r}\right]\frac{C}{r} - \left[\frac{\mathrm{d}^2\alpha_2}{\mathrm{d}r^2}A\right]r\sin\theta - \left[\frac{\mathrm{d}^2\alpha_2}{\mathrm{d}r^2}B\right]r\cos\theta - \frac{\mathrm{d}^2\alpha_2}{\mathrm{d}r^2}C. \end{aligned} \tag{12}$$

Here β_{ij}, α_i are functions of r and N_2', N_3', N_3'', N_4' are differential operators which are defined in Appendix A. Note that a solution that satisfies the system of Eqs. (11) and (12) also satisfies equilibrium equations, generalized Hooke's Law and strain–displacements relations.

2.3. Nonhomogeneous orthotropic axisymmetric case

In this section, the elastostatic problem of a body in the form of a hollow round cylinder of finite length, where the axis of anisotropy coincides with the geometric axis of the body is considered. The tractions acting on the inner and outer surfaces are inner and outer pressures, p_i, p_o, respectively, and the tractions which act on the end surfaces reduce to an axial force, F_z, and to a torsion, T. The inner and outer radii are r_i and r_o as shown in Fig. 2.

Stresses depend only on r due to axisymmetry, and A and B that are multiplying θ terms are set to zero. In the axisymmetric case, $\widehat{F}$, $\hat{\psi}$ are functions of r only, $\widehat{F} = F(r)$, $\hat{\psi} = \psi(r)$. From now on, for simplicity they will be denoted as F and ψ instead of $F(r)$ and $\psi(r)$. As a result of orthotropy, compliances a_{14}, a_{24}, a_{34}, a_{45}, a_{46}, a_{15}, a_{25}, a_{35}, a_{56}, a_{16}, a_{26}, a_{36} are zero. F and ψ in Eqs. (11) and (12) are therefore uncoupled. Eliminating displacement U from ϵ_r and ϵ_θ in Eq. (8) and using Eq. (10) for axisymmetric case, the following equation replaces Eq. (12) for the axisymmetric, orthotropic case:

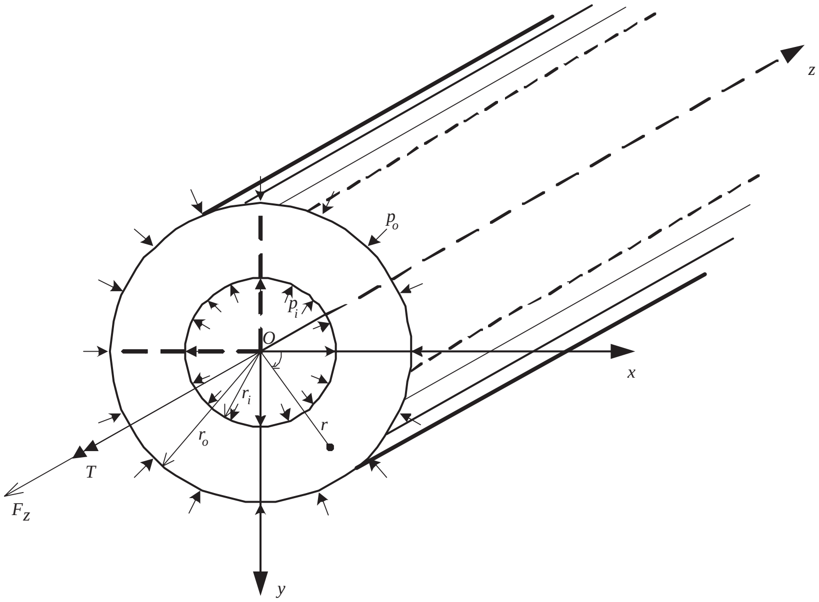


Fig. 2. Axisymmetric cylindrical body.

$$\beta_{22} r \frac{d^3 F}{dr^3} + \left(\beta_{22} + r \frac{d\beta_{22}}{dr}\right) \frac{d^2 F}{dr^2} - \left(\frac{\beta_{11}}{r} - \frac{d\beta_{12}}{dr}\right) \frac{dF}{dr} = \left(\alpha_1 - \alpha_2 - r \frac{d\alpha_2}{dr}\right) C, \tag{13}$$

and similarly, Eq. (7) yields the following to replace Eq. (11):

$$\beta_{44} \frac{d\psi}{dr} = -D_3 r. \tag{14}$$

Boundary conditions on the cylindrical surfaces are:

$$\sigma_r = -p_i \quad \text{at } r = r_i, \tag{15a}$$

$$\sigma_r = -p_o \quad \text{at } r = r_o. \tag{15b}$$

Boundary conditions on the ends are:

$$\int_{r_i}^{r_o} \sigma_z r \, dr = \frac{F_z}{2\pi}, \tag{16a}$$

$$\int_{r_i}^{r_o} \tau_{\theta z} r^2 \, dr = \frac{T}{2\pi}. \tag{16b}$$

2.4. Nondimensionalization of equations

Tractions and lengths are nondimensionalized using a pressure, p and the outer radius, r_o. Dimensionless terms are indicated with an overbar. Nondimensionalized forms of Eqs. (13) and (14) are given as follows:

$$\bar{\beta}_{22} \bar{r} \frac{d^3 \overline{F}}{d\bar{r}^3} + \left(\bar{\beta}_{22} + \bar{r} \frac{d\bar{\beta}_{22}}{d\bar{r}}\right) \frac{d^2 \overline{F}}{d\bar{r}^2} - \left(\frac{\bar{\beta}_{11}}{\bar{r}} - \frac{d\bar{\beta}_{12}}{d\bar{r}}\right) \frac{d\overline{F}}{d\bar{r}} = \left(\alpha_1 - \alpha_2 - \bar{r} \frac{d\alpha_2}{d\bar{r}}\right) C, \tag{17}$$

and

$$\bar{\beta}_{44}\frac{\mathrm{d}\bar{\psi}}{\mathrm{d}\bar{r}} = -\overline{D}_3\bar{r}, \tag{18}$$

dimensionless boundary conditions are:

$$\bar{\sigma}_r = -\frac{p_\mathrm{i}}{p} = -\bar{p}_\mathrm{i} \quad \text{at } \bar{r} = \frac{r_\mathrm{i}}{r_\mathrm{o}} = \bar{r}_\mathrm{i}, \tag{19a}$$

$$\bar{\sigma}_r = -\frac{p_\mathrm{o}}{p} = -\bar{p}_\mathrm{o} \quad \text{at } \bar{r} = 1, \tag{19b}$$

$$\int_{\bar{r}_\mathrm{i}}^{1} \bar{\sigma}_z\bar{r}\,\mathrm{d}\bar{r} = \frac{\overline{F}_z}{2\pi}, \tag{19c}$$

$$\int_{\bar{r}_\mathrm{i}}^{1} \bar{\tau}_{\theta z}\bar{r}^2\,\mathrm{d}\bar{r} = \frac{\overline{T}}{2\pi}, \tag{19d}$$

where $\overline{F}_z$ and $\overline{T}$ are F_z/pr_o^2, T/pr_o^3, respectively.

3. Closed-form solution for nonhomogeneous orthotropic axisymmetric case

Elastic moduli are expressed as power functions of r (Jabbari et al., 2002; Yang, 2000) and their distributions are plotted for positive and negative n in Fig. 3(a) and (b) respectively. Poisson's ratio, v_{ij} is taken constant since the effect of the Poisson's ratio on stresses is small (Kim and Paulino, 2002; Yang, 2000). As a result, using Eq. (9) $\bar{\beta}_{ij}$, α_i can be expressed in terms of engineering coefficients, see Eq. (A.2), as shown below:

$$\begin{aligned}
\bar{\beta}_{11} &= \left(\frac{1}{\overline{E}_{rr}} - \frac{v_{zr}^2}{\overline{E}_{zz}}\right) = \left(\frac{1}{\overline{E}^0_{rr}} - \frac{v_{zr}^2}{\overline{E}^0_{zz}}\right)\frac{1}{\bar{r}^n},\\
\bar{\beta}_{12} &= \left(-\frac{v_{r\theta}}{\overline{E}_{rr}} - \frac{v_{\theta z}v_{zr}}{\overline{E}_{\theta\theta}}\right) = \left(-\frac{v_{r\theta}}{\overline{E}^0_{rr}} - \frac{v_{\theta z}v_{zr}}{\overline{E}^0_{\theta\theta}}\right)\frac{1}{\bar{r}^n},\\
\bar{\beta}_{22} &= \left(\frac{1}{\overline{E}_{\theta\theta}} - \frac{v_{\theta z}^2\overline{E}_{zz}}{(\overline{E}_{\theta\theta})^2}\right) = \left(\frac{1}{\overline{E}^0_{\theta\theta}} - \frac{v_{\theta z}^2\overline{E}^0_{zz}}{(\overline{E}^0_{\theta\theta})^2}\right)\frac{1}{\bar{r}^n},\\
\bar{\beta}_{44} &= \frac{1}{\overline{G}_{\theta z}} = \frac{1}{\overline{G}^0_{\theta z}\bar{r}^n},\\
\alpha_1 &= -v_{zr},\\
\alpha_2 &= -\frac{v_{\theta z}\overline{E}^0_{zz}}{\overline{E}^0_{\theta\theta}}.
\end{aligned} \tag{20}$$

Solution of Eq. (17) for $\overline{F}$ is:

$$\begin{aligned}
\overline{F} = {} & C\frac{4\overline{E}^0_{\theta\theta}(v_{\theta z}\overline{E}^0_{zz} - v_{zr}\overline{E}^0_{\theta\theta})}{(\overline{E}^0_{\theta\theta} - v_{\theta z}^2\overline{E}^0_{zz})((n+2)^2 - \lambda^2)(n+2)}\bar{r}^{n+2} + C_1\left(\frac{2}{n+\lambda+2}\right)\bar{r}^{\left(\frac{n+\lambda}{2}+1\right)}\\
& + C_2\left(\frac{2}{n-\lambda+2}\right)\bar{r}^{\left(\frac{n-\lambda}{2}+1\right)} + C_3,
\end{aligned} \tag{21}$$

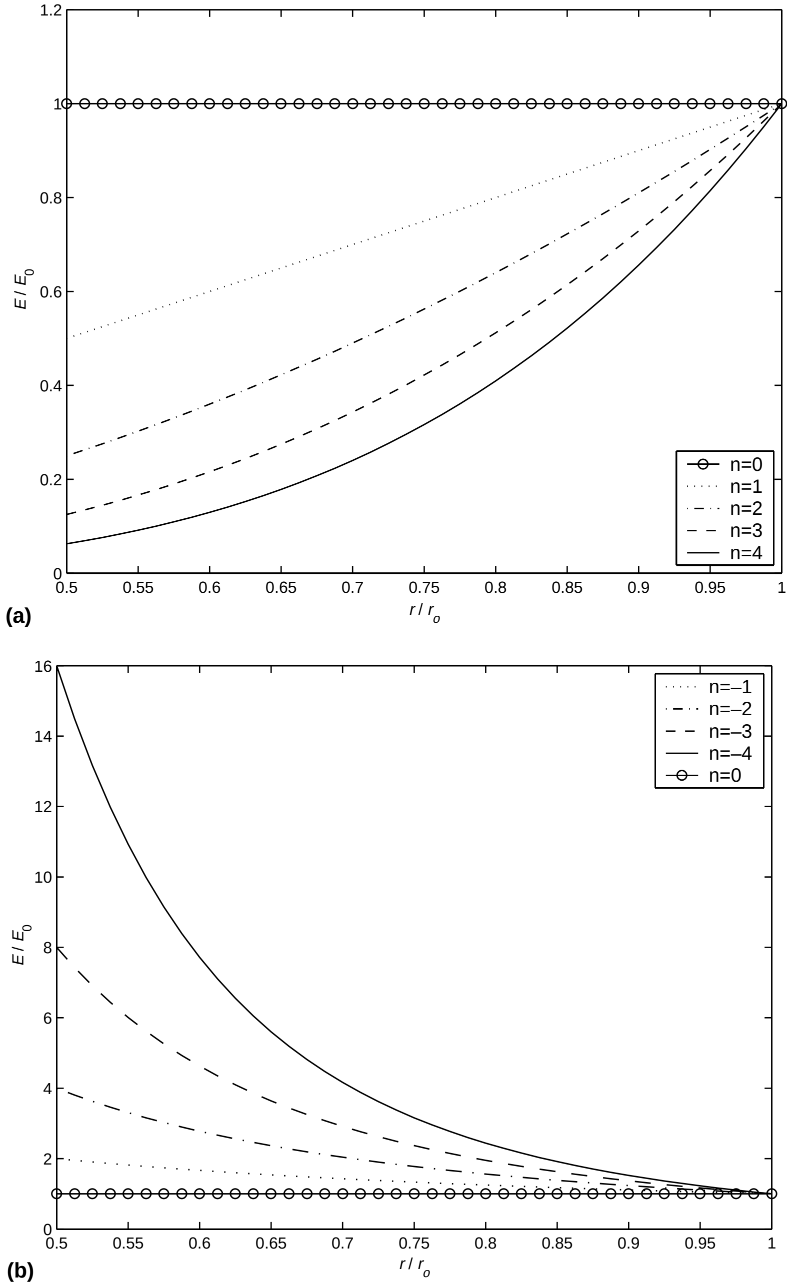


Fig. 3. E/E_0 versus r/r_o, n is (a) positive and (b) negative, $r_i/r_o = 0.5$.

where C_1, C_2 and C_3 are arbitrary constants, λ is:

$$\lambda = \sqrt{\frac{-(n\nu_{\theta z}\overline{E}^0_{zz} + 2\nu_{zr}\overline{E}^0_{\theta\theta})^2\overline{E}^0_{rr} + n^2\overline{E}^0_{rr}\overline{E}^0_{\theta\theta}\overline{E}^0_{zz} + 4(1 - n\nu_{r\theta})(\overline{E}^0_{\theta\theta})^2\overline{E}^0_{zz}}{\overline{E}^0_{rr}\overline{E}^0_{zz}(\overline{E}^0_{\theta\theta} - \nu^2_{\theta z}\overline{E}^0_{zz})}}. \tag{22}$$

Solution of Eq. (18) for $\bar{\psi}$ is:

$$\bar{\psi} = -\frac{\overline{D}_3 \overline{G}^0_{\theta z} \bar{r}^{n+2}}{n+2} + C_4, \tag{23}$$

where C_4 is an arbitrary constant. Deriving stresses from these stress potentials, one gets:

$$\bar{\sigma}_r = C\left[\frac{4\overline{E}^0_{\theta\theta}(v_{\theta z}\overline{E}^0_{zz} - v_{zr}\overline{E}^0_{\theta\theta})}{\overline{E}^0_{\theta\theta} - v^2_{\theta z}\overline{E}^0_{zz}}\right]\frac{\bar{r}^n}{(n+2)^2 - \lambda^2} + C_1\bar{r}^{\left(\frac{n+\lambda}{2}-1\right)} + C_2\bar{r}^{\left(\frac{n-\lambda}{2}-1\right)}, \tag{24a}$$

$$\bar{\sigma}_\theta = C\left[\frac{4(n+1)\overline{E}^0_{\theta\theta}(v_{\theta z}\overline{E}^0_{zz} - v_{zr}\overline{E}^0_{\theta\theta})}{\overline{E}^0_{\theta\theta} - v^2_{\theta z}\overline{E}^0_{zz}}\right]\frac{\bar{r}^n}{(n+2)^2 - \lambda^2} + C_1\left(\frac{n+\lambda}{2}\right)\bar{r}^{\left(\frac{n+\lambda}{2}-1\right)} + C_2\left(\frac{n-\lambda}{2}\right)\bar{r}^{\left(\frac{n-\lambda}{2}-1\right)}, \tag{24b}$$

$$\begin{aligned}\bar{\sigma}_z = {} & C\left[\overline{E}^0_{zz}((n+2)^2 - \lambda^2) + \frac{4v_{zr}\overline{E}^0_{\theta\theta}(v_{\theta z}\overline{E}^0_{zz} - v_{zr}\overline{E}^0_{\theta\theta})}{\overline{E}^0_{\theta\theta} - v^2_{\theta z}\overline{E}^0_{zz}} + \frac{4(n+1)v_{\theta z}\overline{E}^0_{zz}(v_{\theta z}\overline{E}^0_{zz} - v_{zr}\overline{E}^0_{\theta\theta})}{\overline{E}^0_{\theta\theta} - v^2_{\theta z}\overline{E}^0_{zz}}\right]\frac{\bar{r}^n}{(n+2)^2 - \lambda^2} \\ & + C_1\left[v_{zr} + v_{\theta z}\frac{(n+\lambda)\overline{E}^0_{zz}}{2\overline{E}^0_{\theta\theta}}\right]\bar{r}^{\left(\frac{n+\lambda}{2}-1\right)} + C_2\left[v_{zr} + v_{\theta z}\frac{(n-\lambda)\overline{E}^0_{zz}}{2\overline{E}^0_{\theta\theta}}\right]\bar{r}^{\left(\frac{n-\lambda}{2}-1\right)},\end{aligned} \tag{24c}$$

and

$$\bar{\tau}_{\theta z} = \bar{\phi}\overline{G}^0_{\theta z}\bar{r}^{n+1}. \tag{24d}$$

C, C_1, C_2 and $\bar{\phi}$ (note $\overline{D}_3 = \bar{\phi}$) can be obtained from boundary conditions and these are given in Appendix A. $\bar{\phi}$ is the relative angle of twist (Lekhnitskii, 1963).

4. Results

In the following examples analytical stress solutions are verified using FE method, different degrees of material nonhomogeneity are studied by varying n and torsional load effect on stresses is analyzed.

4.1. Example 1: verification using FE method

A nonhomogeneous, orthotropic hollow cylinder loaded by $\bar{p}_i = 10$, $\bar{p}_o = 1$, $\overline{T} = 5$ and $\overline{F}_z = 12$ is studied. Elastic properties are: $\overline{E}_{rr} = 5 \times 10^5\bar{r}^2$, $\overline{E}_{\theta\theta} = 7 \times 10^5\bar{r}^2$, $\overline{E}_{zz} = 6 \times 10^5\bar{r}^2$, $\overline{G}_{\theta z} = 1.5 \times 10^5\bar{r}^2$, $v_{\theta z} = 0.35$, $v_{z\,r} = 0.2$, $v_{r\theta} = 0.27$. Dimensionless inner radius is $\bar{r}_i = 0.5$. Stresses are calculated from Eqs. (24a)–(24d) as follows:

$$\bar{\sigma}_r = 0.2909\bar{r}^2 + 3.6741\bar{r}^{1.23} - 4.9650\bar{r}^{-1.23}, \tag{25a}$$

$$\bar{\sigma}_\theta = 0.8726\bar{r}^2 + 8.1904\bar{r}^{1.23} + 1.1381\bar{r}^{-1.23}, \tag{25b}$$

$$\bar{\sigma}_z = 5.8739\bar{r}^2 + 3.1919\bar{r}^{1.23} - 0.6516\bar{r}^{-1.23} \tag{25c}$$

and

$$\bar{\tau}_{\theta z} = 4.8504\bar{r}^3. \tag{25d}$$

The results shown above are plotted with FE results in Figs. 4–6. In these figures, ‘‘analytical’’ refers to closed-form results obtained in this study. Using symmetry boundary conditions 1/8 of the hollow cylinder

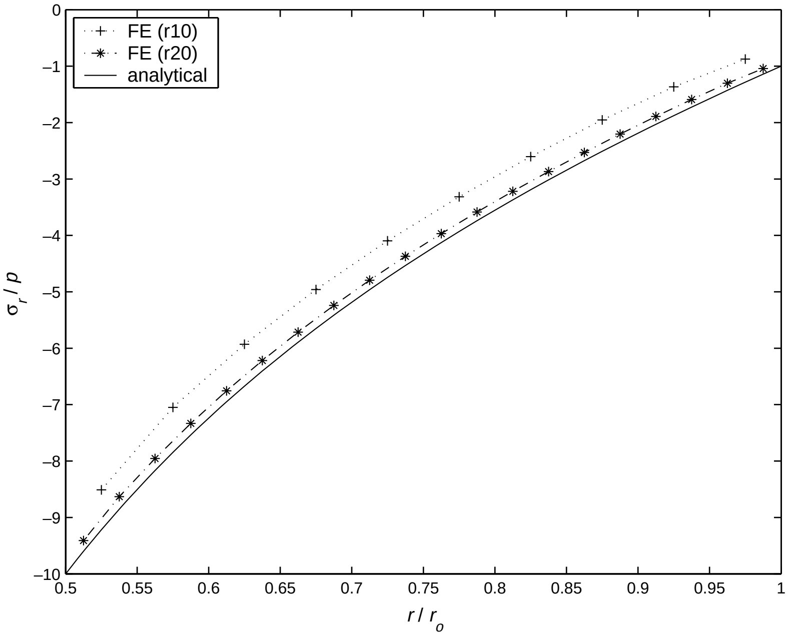


Fig. 4. σ_r/p versus r/r_o where $n = 2$, $\bar{r}_i = 0.5$, $\bar{p}_i = 10$, $\bar{p}_o = 1$ and $\overline{F}_z = 12$.

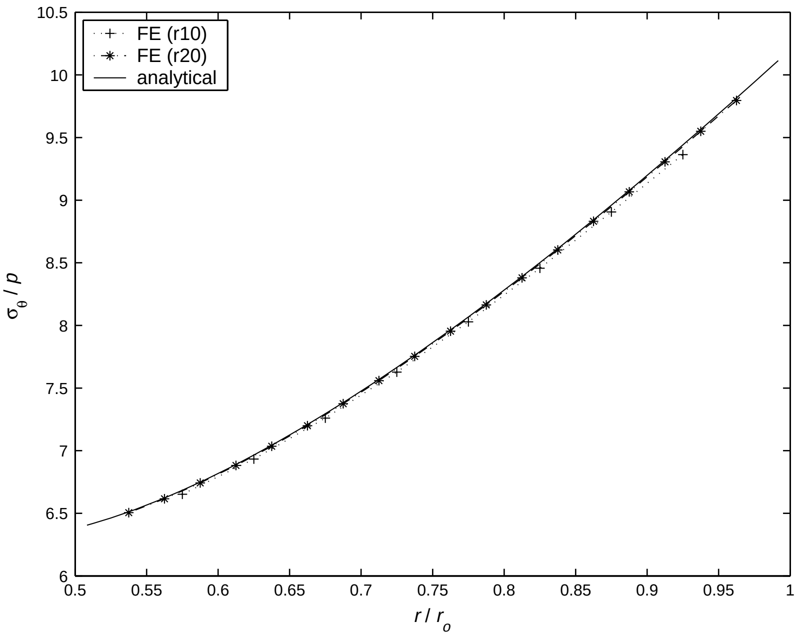


Fig. 5. σ_θ/p versus r/r_o where $n = 2$, $\bar{r}_i = 0.5$, $\bar{p}_i = 10$, $\bar{p}_o = 1$ and $\overline{F}_z = 12$.

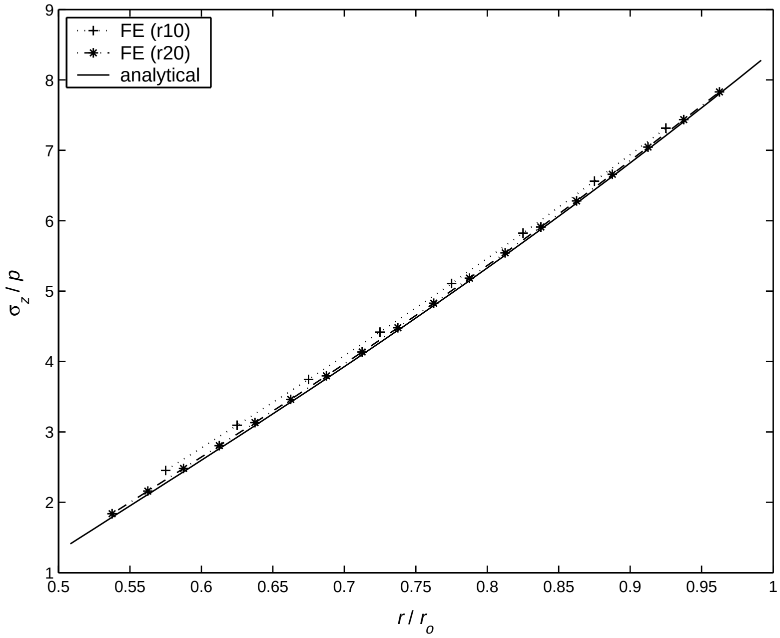


Fig. 6. σ_z/p versus r/r_o where $n = 2$, $\bar{r}_i = 0.5$, $\bar{p}_i = 10$, $\bar{p}_o = 1$ and $\overline{F}_z = 12$.

is modelled with PATRAN and is solved with NASTRAN. FE(r10) and FE(r20) are finite element models that contain 5000 elements with 23441 nodes (about 66740 dof) and 20000 elements with 87381 nodes (about 254580 dof), respectively. Figs. 4–6 show that results are in good agreement with FE results.

4.2. Example 2: different degree of material nonhomogeneity varying n

As a second case, only a uniform inner pressure is applied as shown in Fig. 7 where $\bar{p}_i = 1$. The axial and the torsional loads are not present either. Nonhomogeneity is along the r-direction, and it is in the form of a power law, r^n. When n is positive, the inner part of the cylinder has a smaller elastic modulus than that of the outer part, the contrary is the case when n is negative. When n is equal to 0, the cylinder is homogeneous.

Dimensionless engineering properties are the same as the previous example. $\bar{\sigma}_r$, $\bar{\sigma}_\theta$ and $\bar{\sigma}_z$ are plotted in Figs. 8(a), 9(a) and 10(a) for positive n and in Figs. 8(b), 9(b) and 10(b) for negative n. $\bar{\sigma}_r$ for positive and negative n are shown in Figs. 8(a) and 8(b), $\bar{\sigma}_r$ is compressive and has smaller absolute values for negative n. This means that when the inner part has a larger elastic modulus, $\bar{\sigma}_r$ decreases. On the other hand, in Fig. 9(a) for positive n, $\bar{\sigma}_\theta$ is tensile and its value increases for decreasing values of n at the inner part, it decreases for decreasing values of n at the outer part. A similar behavior is observed for negative n in Fig. 9(b). Maximum value of hoop stress occurs at the inner part for $n = -4$. Hoop stress, $\bar{\sigma}_\theta$, is higher at the inner wall and decreases toward the outer wall in the homogeneous case. For negative values of n, the amplitude of $\bar{\sigma}_\theta$ at the inner wall increases but its behavior through the wall thickness does not change (Fig. 9(b)). For positive values of n however, from $n = 2$ and on, the inner wall value of the hoop stress is less than the outer wall value, and it increases through the wall thickness (Fig. 9(a)) contrary to the homogeneous case. In Fig. 10(a) for positive n, $\bar{\sigma}_z$ changes from compressive to tensile on the inner boundary, and changes from tensile to compressive on the outer boundary for decreasing values of n. In Fig. 10(b) where n

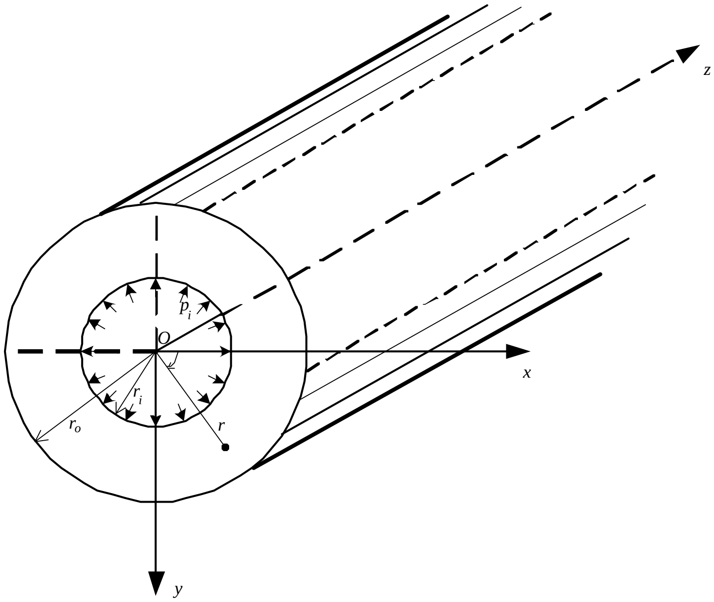


Fig. 7. A pressurized thick walled cylinder.

is negative, $\bar{\sigma}_z$ is tensile and its value increases with the decreasing values of n at the inner part, but it is compressive, and its absolute value decreases for the decreasing values of n. For the homogeneous case, $\bar{\sigma}_z$ is tensile at the inner part and compressive at the outer part. Values of $\bar{\sigma}_z$ are relatively small when compared to $\bar{\sigma}_r$ and $\bar{\sigma}_\theta$. The dimensionless Von Mises stress, $\bar{\sigma}_{\text{eff}}$ is:

$$\bar{\sigma}_{\text{eff}} = \sqrt{\frac{(\bar{\sigma}_r - \bar{\sigma}_\theta)^2 + (\bar{\sigma}_\theta - \bar{\sigma}_z)^2 + (\bar{\sigma}_z - \bar{\sigma}_r)^2 + 6(\bar{\tau}_{\theta z} + \bar{\tau}_{rz} + \bar{\tau}_{r\theta})}{2}}. \tag{26}$$

Values of n are varied between −4 and 4, and $\bar{\sigma}_{\text{eff}}$ is plotted in Fig. 11. For $n = -4$ the inner part of the cylinder is more critical than the outer part and for $n = 4$ the outer part the cylinder is more critical than the inner part due to the larger $\bar{\sigma}_{\text{eff}}$. An n value that equates values of $\bar{\sigma}_{\text{eff}}$ at the inner and the outer parts of the cylinder provides an optimum situation since values of $\bar{\sigma}_{\text{eff}}$ is less than the values at the boundaries that means it has a convex behavior for a given n. For this example, when $n = 2.39$, $\bar{\sigma}_{\text{eff}}$ is equal both at the inner and the outer parts of the cylinder as shown in Fig. 12.

4.3. Example 3: torsional effect

Third example will be the addition of a torque $\overline{T} = 5$ to example 2. $\bar{\sigma}_r$, $\bar{\sigma}_\theta$ and $\bar{\sigma}_z$ will be the same, because there is no effect of torque on normal stresses. $\bar{\tau}_{\theta z}$ is plotted in Fig. 13(a) for positive n and in Fig. 13(b) for negative n. For the homogeneous case, $n = 0$, $\bar{\tau}_{\theta z}$ is linear. On the other hand, an interesting situation is that $\tau_{\theta z}$ has a constant value along the radial direction for $n = -1$. This also indicates that the body has a constant relative angle of twist, ϕ for $n = -1$. (Note that $\bar{\tau}_{\theta z} = \bar{\phi}\overline{G}^0_{\theta z}\bar{r}^{n+1}$ and when $n = -1$, $\bar{r}^{n+1}$ becomes 1.)

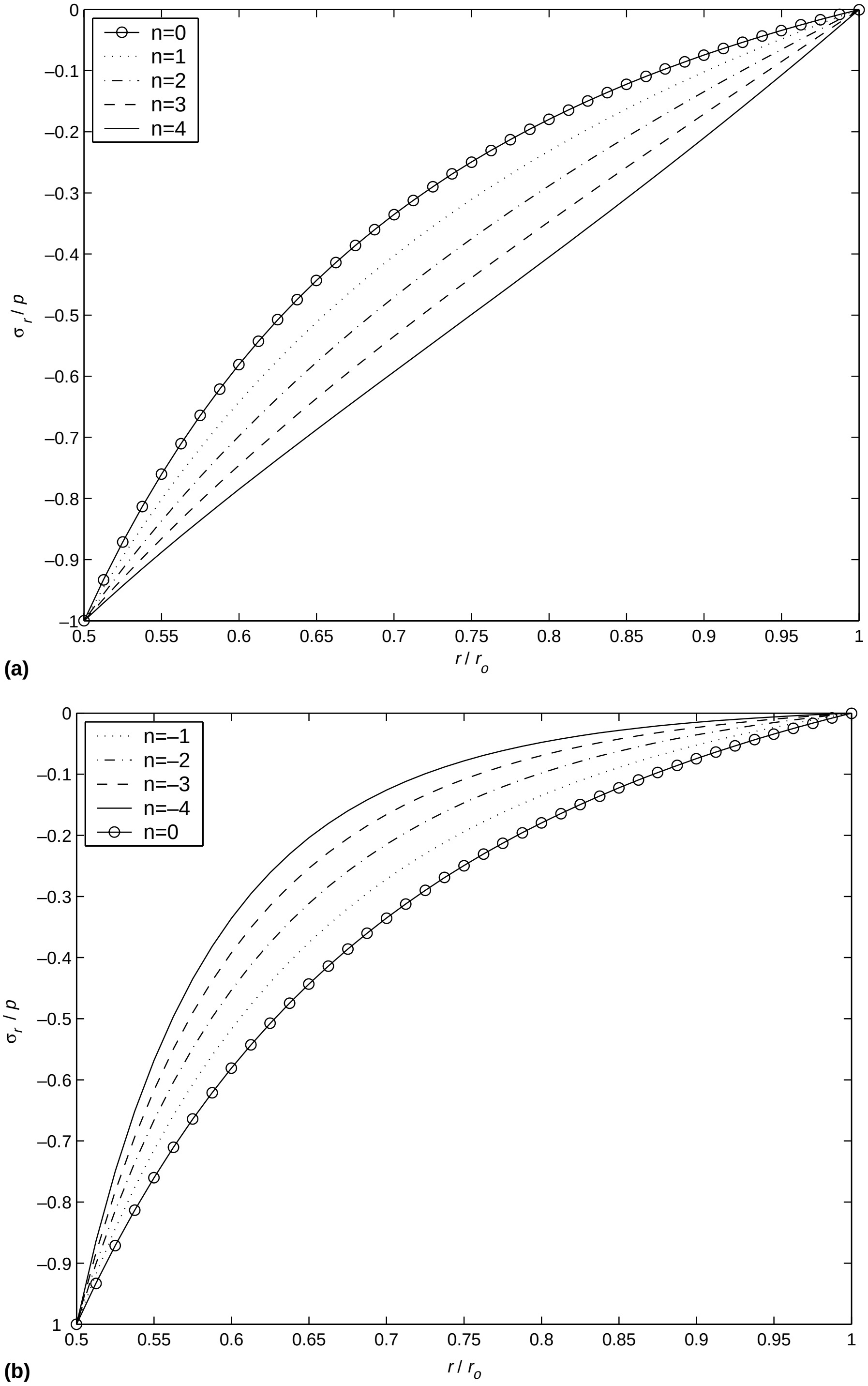


Fig. 8. σ_r/p versus r/r_o, n is (a) positive and (b) negative, $r_i/r_o = 0.5$ and $p_i/p = 1$.

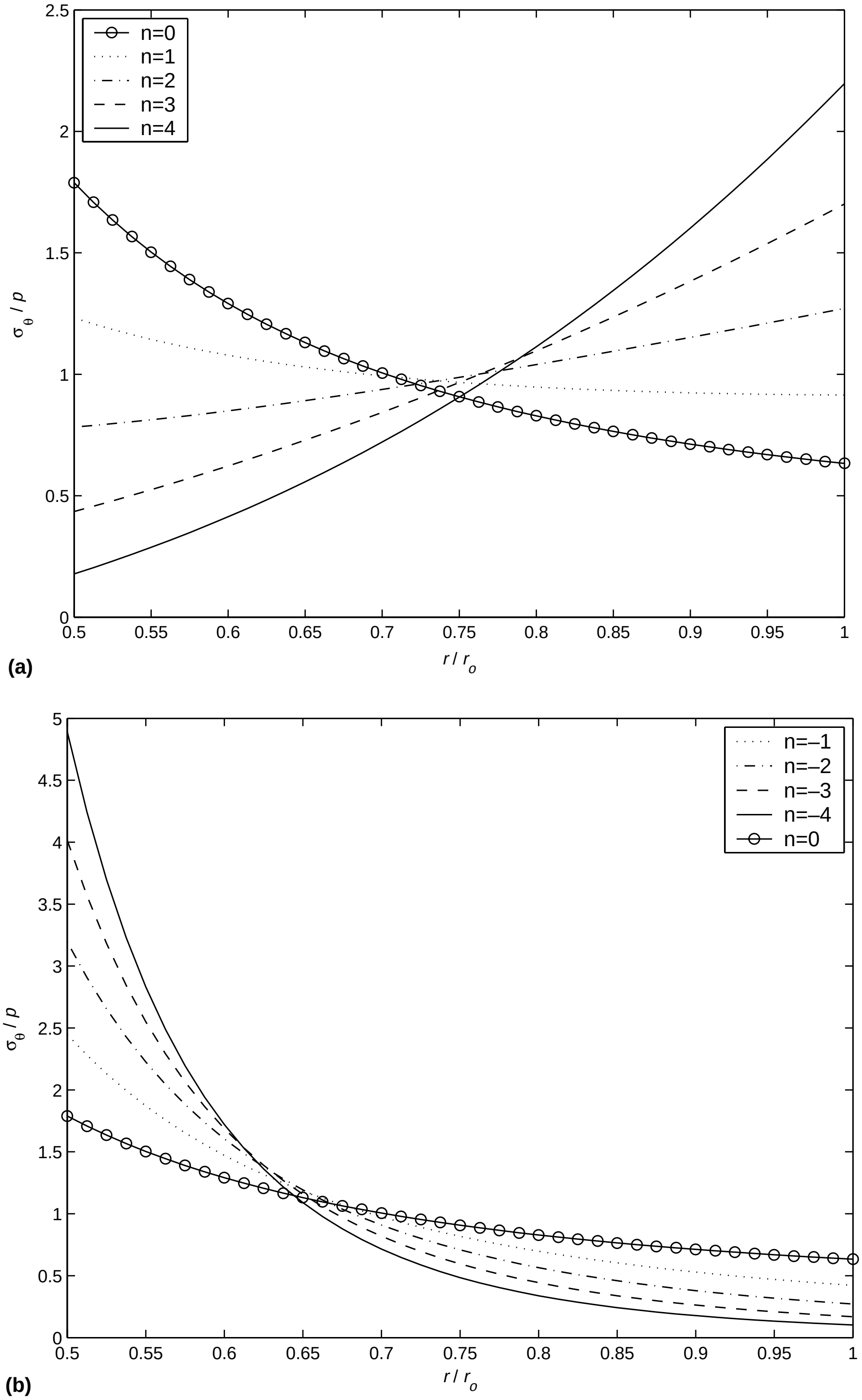
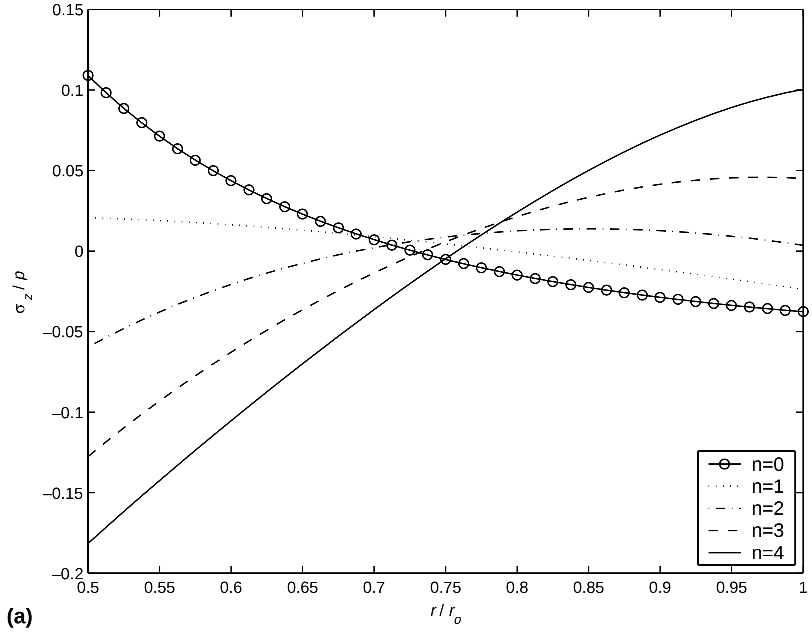


Fig. 9. σ_θ/p versus r/r_o, n is (a) positive and (b) negative, $r_i/r_o = 0.5$ and $p_i/p = 1$.



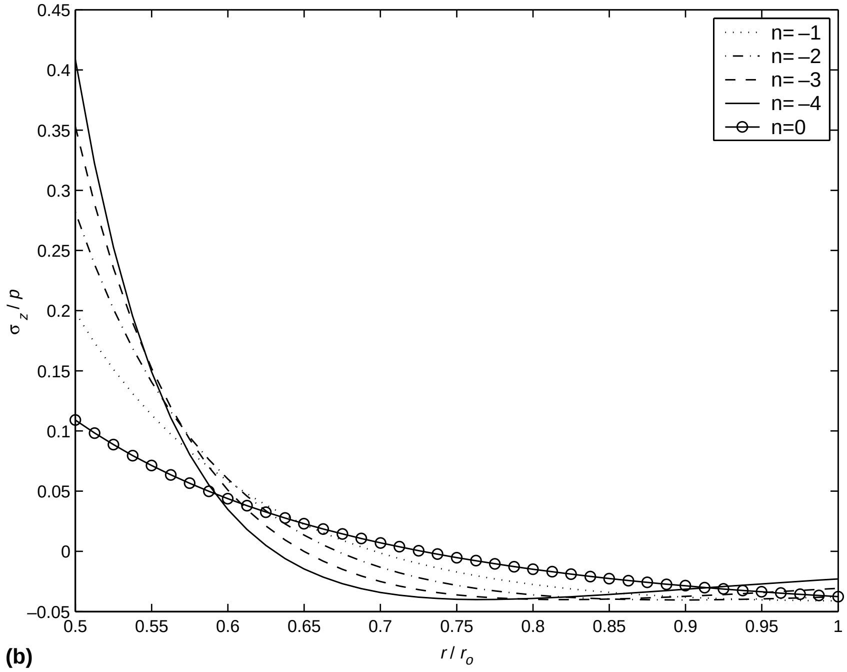


Fig. 10. σ_z/p versus r/r_o, n is (a) positive and (b) negative, $r_i/r_o = 0.5$ and $p_i/p = 1$.

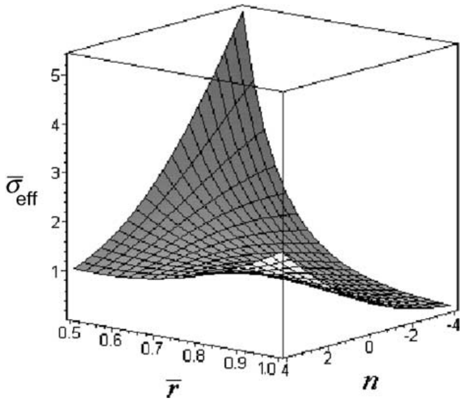


Fig. 11. Von Mises stress, $\bar{\sigma}_{eff}$ versus $\bar{r}$ and n, $\bar{r}_i = 0.5$ and $\bar{p}_i = 1$.

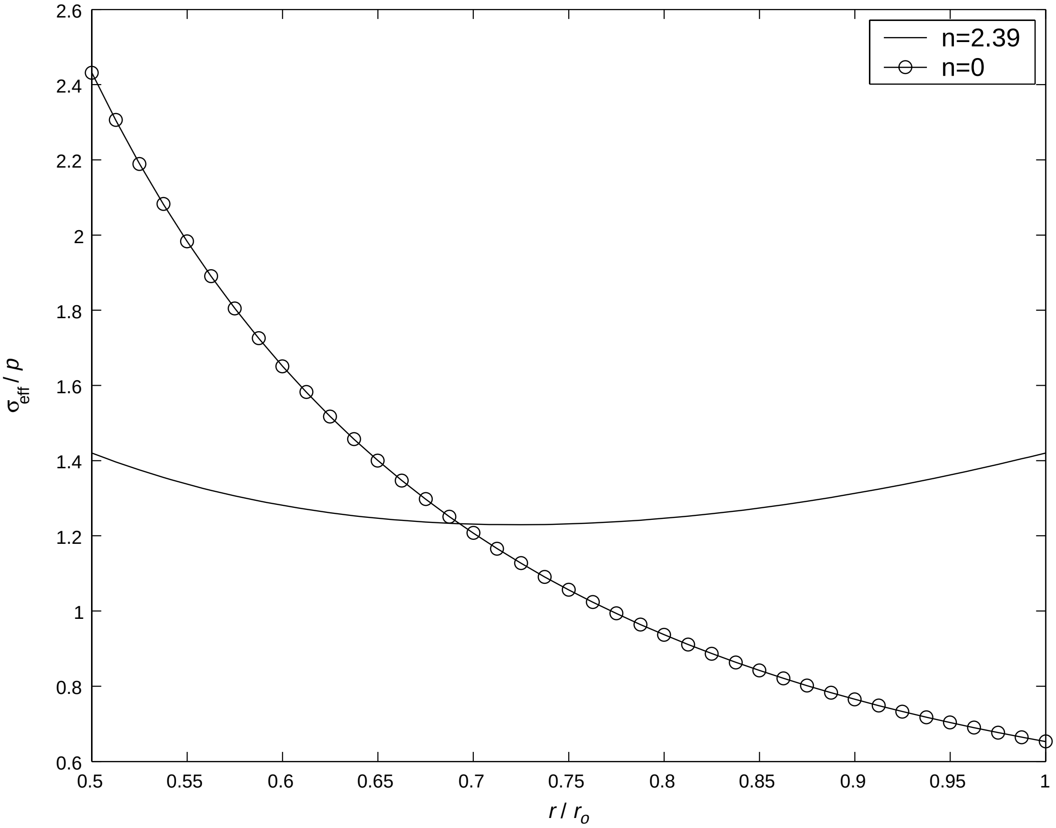


Fig. 12. Von Mises stress, $\bar{\sigma}_{eff}$ versus $\bar{r}$ for $n = 2.39$ and $n = 0$ (*homogeneous*), $\bar{r}_i = 0.5$ and $\bar{p}_i = 1$.

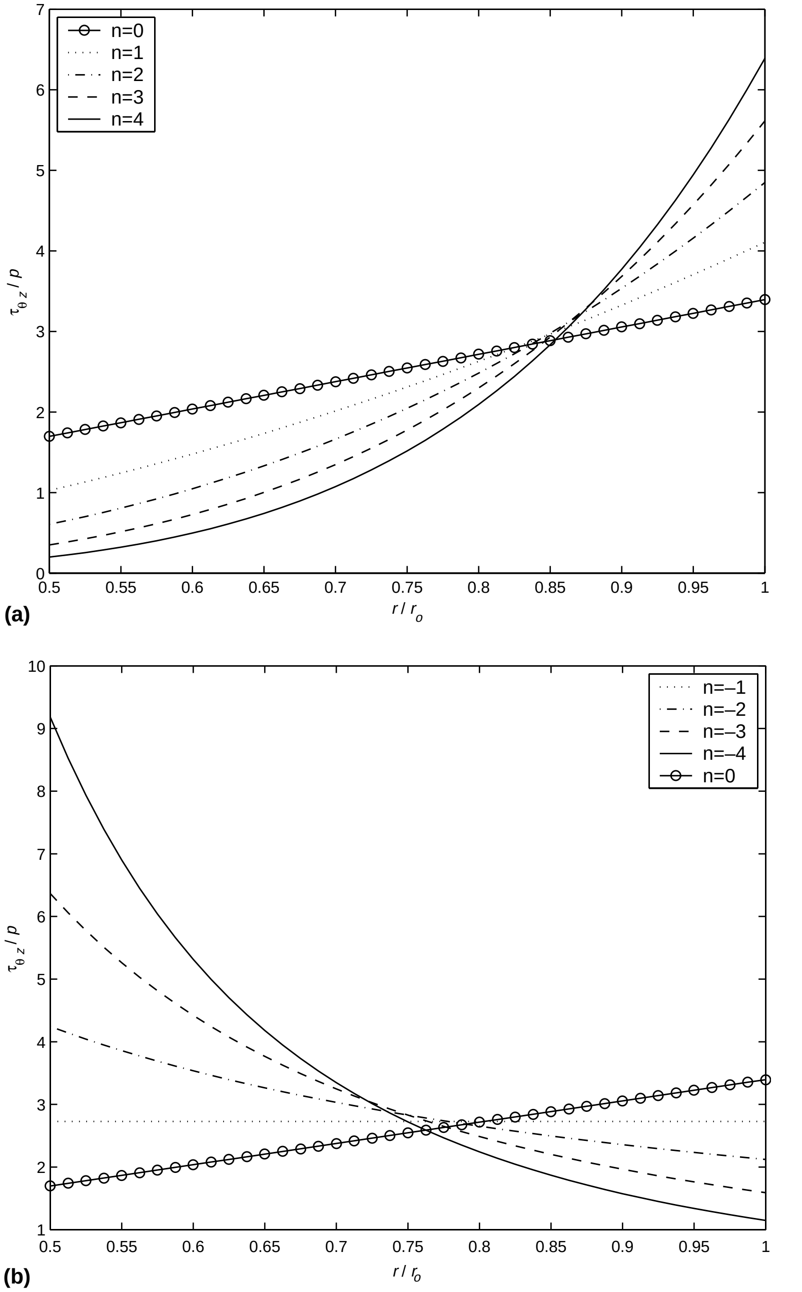


Fig. 13. $\tau_{\theta z}/p$ versus r/r_o, n is (a) positive and (b) negative, $r_i/r_o = 0.5$, $p_i/p = 1$ and $\overline{T} = 5$.

5. Concluding remarks

In this study, effects of radially varying moduli on the stress distribution of nonhomogeneous anisotropic cylindrical bodies are investigated. A system of equations is obtained for the most general nonhomogeneous fully anisotropic cylindrical body. Closed-form stress solutions for nonhomogeneous orthotropic axisymmetric cylindrical body are obtained with Young's and shear moduli varying according to a power law, r^n. Results are verified using a three-dimensional finite element model. A pressurized thick walled cylinder is studied in detail. Stresses for various values of n are obtained including the homogeneous case ($n = 0$).

Although very difficult and expensive to manufacture, above results show that FGM systems can be designed for various purposes of applications. Material distribution can be tailored to obtain a desired stress distribution.

Acknowledgments

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Appendix A

Generalized Hooke's law is:

$$\begin{bmatrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix}. \tag{A.1}$$

Orthotropic compliance:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}, \tag{A.2a}$$

where

$$\begin{aligned} &a_{11} = \frac{1}{E_{rr}}, \quad a_{22} = \frac{1}{E_{\theta\theta}}, \quad a_{33} = \frac{1}{E_{zz}}, \\ &a_{12} = -\frac{\nu_{r\theta}}{E_{rr}}, \quad a_{13} = -\frac{\nu_{zr}}{E_{zz}}, \quad a_{23} = -\frac{\nu_{\theta z}}{E_{\theta\theta}}, \\ &a_{44} = \frac{1}{G_{\theta z}}, \quad a_{55} = \frac{1}{G_{rz}}, \quad a_{66} = \frac{1}{G_{r\theta}}, \end{aligned} \tag{A.2b}$$

Differential operators for the system of Eqs. (12) and (11) are as follows:

$$N_2' = \beta_{44}\frac{\partial^2}{\partial r^2} - \frac{2\beta_{45}}{r}\frac{\partial^2}{\partial r\partial\theta} + \beta_{55}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \left(\frac{\beta_{44}}{r} + \frac{\mathrm{d}\beta_{44}}{\mathrm{d}r}\right)\frac{\partial}{\partial r} - \frac{\mathrm{d}\beta_{45}}{\mathrm{d}r}\frac{1}{r}\frac{\partial}{\partial\theta}, \tag{A.3a}$$

$$\begin{aligned} N_3' = {} & -\beta_{24}\frac{\partial^3}{\partial r^3} + \left(\frac{\beta_{25}}{r} + \frac{\beta_{46}}{r}\right)\frac{\partial^3}{\partial r^2\partial\theta} - \left(\frac{\beta_{14}}{r^2} + \frac{\beta_{56}}{r^2}\right)\frac{\partial^3}{\partial r\partial\theta^2} + \frac{\beta_{15}}{r^3}\frac{\partial^3}{\partial\theta^3} + \left(\frac{\beta_{14}}{r} - \frac{2\beta_{24}}{r} - 2\frac{\mathrm{d}\beta_{24}}{\mathrm{d}r}\right)\frac{\partial^2}{\partial r^2} \\ & - \left(\frac{\beta_{15}}{r^2} - \frac{\beta_{46}}{r^2} - \frac{2}{r}\frac{\mathrm{d}\beta_{25}}{\mathrm{d}r} - \frac{1}{r}\frac{\mathrm{d}\beta_{46}}{\mathrm{d}r}\right)\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\mathrm{d}\beta_{56}}{\mathrm{d}r}\frac{\partial^2}{\partial\theta^2} + \left(\frac{1}{r}\frac{\mathrm{d}\beta_{14}}{\mathrm{d}r} - \frac{2}{r}\frac{\mathrm{d}\beta_{24}}{\mathrm{d}r} - \frac{\mathrm{d}^2\beta_{24}}{\mathrm{d}r^2}\right)\frac{\partial}{\partial r} \\ & + \left(\frac{\beta_{15}}{r^3} - \frac{1}{r^2}\frac{\mathrm{d}\beta_{15}}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}^2\beta_{25}}{\mathrm{d}r^2}\right)\frac{\partial}{\partial\theta}, \end{aligned} \tag{A.3b}$$

$$\begin{aligned} N_3'' = {} & -\beta_{24}\frac{\partial^3}{\partial r^3} + \left(\frac{\beta_{25}}{r} + \frac{\beta_{46}}{r}\right)\frac{\partial^3}{\partial r^2\partial\theta} - \left(\frac{\beta_{14}}{r^2} + \frac{\beta_{56}}{r^2}\right)\frac{\partial^3}{\partial r\partial\theta^2} + \frac{\beta_{15}}{r^3}\frac{\partial^3}{\partial\theta^3} - \left(\frac{\beta_{14}}{r} + \frac{\beta_{24}}{r} + \frac{\mathrm{d}\beta_{24}}{\mathrm{d}r}\right)\frac{\partial^2}{\partial r^2} \\ & + \left(\frac{\beta_{15}}{r^2} - \frac{\beta_{46}}{r^2} + \frac{1}{r}\frac{\mathrm{d}\beta_{46}}{\mathrm{d}r}\right)\frac{\partial^2}{\partial r\partial\theta} + \left(\frac{\beta_{14}}{r^3} + \frac{\beta_{56}}{r^3} - \frac{1}{r^2}\frac{\mathrm{d}\beta_{14}}{\mathrm{d}r}\right)\frac{\partial^2}{\partial\theta^2} \\ & - \frac{1}{r}\frac{\mathrm{d}\beta_{14}}{\mathrm{d}r}\frac{\partial}{\partial r} + \left(\frac{\beta_{46}}{r^3} - \frac{1}{r^2}\frac{\mathrm{d}\beta_{46}}{\mathrm{d}r}\right)\frac{\partial}{\partial\theta}, \end{aligned} \tag{A.3c}$$

$$\begin{aligned} N_4' = {} & \beta_{22}\frac{\partial^4}{\partial r^4} - \frac{2\beta_{26}}{r}\frac{\partial^4}{\partial r^3\partial\theta} + \left(\frac{2\beta_{12}}{r^2} + \frac{\beta_{66}}{r^2}\right)\frac{\partial^4}{\partial r^2\partial\theta^2} - \frac{2\beta_{16}}{r^3}\frac{\partial^4}{\partial r\partial\theta^3} + \frac{\beta_{11}}{r^4}\frac{\partial^4}{\partial\theta^4} + \left(\frac{2\beta_{22}}{r} + 2\frac{\mathrm{d}\beta_{22}}{\mathrm{d}r}\right)\frac{\partial^3}{\partial r^3} \\ & - \frac{3}{r}\frac{\mathrm{d}\beta_{26}}{\mathrm{d}r}\frac{\partial^3}{\partial r^2\partial\theta} - \left(\frac{2\beta_{12}}{r^3} + \frac{\beta_{66}}{r^3} - \frac{2}{r^2}\frac{\mathrm{d}\beta_{12}}{\mathrm{d}r} - \frac{1}{r^2}\frac{\mathrm{d}\beta_{66}}{\mathrm{d}r}\right)\frac{\partial^3}{\partial r\partial\theta^2} + \left(\frac{2\beta_{16}}{r^4} - \frac{1}{r^3}\frac{\mathrm{d}\beta_{16}}{\mathrm{d}r}\right)\frac{\partial^3}{\partial\theta^3} \\ & - \left(\frac{\beta_{11}}{r^2} - \frac{1}{r}\frac{\mathrm{d}\beta_{12}}{\mathrm{d}r} - \frac{2}{r}\frac{\mathrm{d}\beta_{22}}{\mathrm{d}r} - \frac{\mathrm{d}^2\beta_{22}}{\mathrm{d}r^2}\right)\frac{\partial^2}{\partial r^2} - \left(\frac{2\beta_{16}}{r^3} + \frac{2\beta_{26}}{r^3} - \frac{2}{r^2}\frac{\mathrm{d}\beta_{26}}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}^2\beta_{26}}{\mathrm{d}r^2}\right)\frac{\partial^2}{\partial r\partial\theta} \\ & + \left(\frac{2\beta_{11}}{r^4} + \frac{2\beta_{12}}{r^4} + \frac{\beta_{66}}{r^4} - \frac{1}{r^3}\frac{\mathrm{d}\beta_{11}}{\mathrm{d}r} - \frac{2}{r^3}\frac{\mathrm{d}\beta_{12}}{\mathrm{d}r} - \frac{1}{r^3}\frac{\mathrm{d}\beta_{66}}{\mathrm{d}r} + \frac{1}{r^2}\frac{\mathrm{d}^2\beta_{12}}{\mathrm{d}r^2}\right)\frac{\partial^2}{\partial\theta^2} \\ & + \left(\frac{\beta_{11}}{r^3} - \frac{1}{r^2}\frac{\mathrm{d}\beta_{11}}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}^2\beta_{12}}{\mathrm{d}r^2}\right)\frac{\partial}{\partial r} + \left(\frac{2\beta_{16}}{r^4} + \frac{2\beta_{26}}{r^4} - \frac{1}{r^3}\frac{\mathrm{d}\beta_{16}}{\mathrm{d}r} - \frac{2}{r^3}\frac{\mathrm{d}\beta_{26}}{\mathrm{d}r} + \frac{1}{r^2}\frac{\mathrm{d}^2\beta_{26}}{\mathrm{d}r^2}\right)\frac{\partial}{\partial\theta}. \end{aligned} \tag{A.3d}$$

Integrating constants, C, C_1, C_2 and obtained from boundary conditions are:

$$\begin{aligned} C = {} & \Big[\overline{E}^0_{\theta\theta}\Big\{4\pi\bar{p}_\mathrm{i}\Big((n+2)(n+\lambda+2)g_{-\lambda}\bar{r}_\mathrm{i}^{(n+2)} - (n+2)(n-\lambda+2)g_\lambda\bar{r}_\mathrm{i}^{(n+\lambda+2)} + ((n+2)(n-\lambda+2)g_\lambda \\ & - (n+2)(n+\lambda+2)g_{-\lambda})\bar{r}_\mathrm{i}^{\left(\frac{n+\lambda}{2}+1\right)}\Big) + 4\pi\bar{p}_\mathrm{o}\Big((n+2)(n+\lambda+2)g_{-\lambda}\bar{r}_\mathrm{i}^{(n+\lambda)} - (n+2)(n-\lambda+2)g_\lambda\bar{r}_\mathrm{i}^{n} \\ & + ((n+2)(n-\lambda+2)g_\lambda - (n+2)(n+\lambda+2)g_{-\lambda})\bar{r}_\mathrm{i}^{\left(\frac{3n+\lambda}{2}+1\right)}\Big) \\ & + \overline{F}_z\Big((n+2)(n+\lambda+2)(n-\lambda+2)(\bar{r}_\mathrm{i}^{(n+\lambda)} - \bar{r}_\mathrm{i}^{n})\Big)\Big\}\Big]\Big/K, \end{aligned} \tag{A.4a}$$

$$C_1 = \Big[2\pi\bar{p}_i\Big(2(n+2)(n+\lambda+2)\overline{E}^0_{\theta\theta}\mu g_{-\lambda}\Big(\bar{r}_i^{\left(\frac{n+\lambda}{2}+1\right)} - \bar{r}_i^{(n+2)}\Big)$$
$$+\Big((n+1)(n+\lambda+2)(n-\lambda+2)v_{\theta z}\overline{E}^0_{zz}\mu$$
$$+((n+2)^2-\lambda^2)\overline{E}^0_{\theta\theta}(\overline{E}^0_{zz}+v_{zr}\mu)\Big)\Big(\bar{r}_i^{\left(\frac{3n+\lambda}{2}+3\right)} - \bar{r}_i^{\left(\frac{n+\lambda}{2}+1\right)}\Big)\Big)$$
$$+\ 2\pi\bar{p}_o\Big(2(n+2)(n+\lambda+2)\overline{E}^0_{\theta\theta}\mu g_{-\lambda}\Big(\bar{r}_i^{(2n+2)} - \bar{r}_i^{\left(\frac{3n+\lambda}{2}+1\right)}\Big)$$
$$+\Big((n+1)(n+\lambda+2)(n-\lambda+2)v_{\theta z}\overline{E}^0_{zz}\mu + ((n+2)^2-\lambda^2)\overline{E}^0_{\theta\theta}(\overline{E}^0_{zz}+v_{zr}\mu)\Big)\Big(\bar{r}_i^n - \bar{r}_i^{(2n+2)}\Big)\Big)$$
$$+\overline{F}_z\Big((n+2)(n+\lambda+2)(n-\lambda+2)\overline{E}^0_{\theta\theta}\mu\Big(\bar{r}_i^n - \bar{r}_i^{\left(\frac{3n+\lambda}{2}+1\right)}\Big)\Big)\Big]\Big/K, \qquad \text{(A.4b)}$$

$$C_2 = \Big[2\pi\bar{p}_i\Big(2(n+2)(n-\lambda+2)\overline{E}^0_{\theta\theta}\mu g_{\lambda}\Big(\bar{r}_i^{(n+\lambda+2)} - \bar{r}_i^{\left(\frac{n+\lambda}{2}+1\right)}\Big)$$
$$+\Big((n+1)(n+\lambda+2)(n-\lambda+2)v_{\theta z}\overline{E}^0_{zz}\mu + ((n+2)^2-\lambda^2)\overline{E}^0_{\theta\theta}(\overline{E}^0_{zz}+v_{zr}\mu)\Big)\Big(\bar{r}_i^{\left(\frac{n+\lambda}{2}+1\right)} - \bar{r}_i^{\left(\frac{3n+\lambda}{2}+3\right)}\Big)\Big)$$
$$+2\pi\bar{p}_o\Big(2(n+2)(n-\lambda+2)\overline{E}^0_{\theta\theta}\mu g_{\lambda}\Big(\bar{r}_i^{\left(\frac{3n+\lambda}{2}+1\right)} - \bar{r}_i^{(2n+\lambda+2)}\Big) + \Big((n+1)(n+\lambda+2)(n-\lambda+2)v_{\theta z}\overline{E}^0_{zz}\mu$$
$$+((n+2)^2-\lambda^2)\overline{E}^0_{\theta\theta}(\overline{E}^0_{zz}+v_{zr}\mu)\Big)\Big(\bar{r}_i^{(2n+\lambda+2)} - \bar{r}_i^{(n+\lambda)}\Big)\Big)$$
$$+\overline{F}_z\Big((n+2)(n+\lambda+2)(n-\lambda+2)\overline{E}^0_{\theta\theta}\mu\Big(\bar{r}_i^{\left(\frac{3n+\lambda}{2}+1\right)} - \bar{r}_i^{(n+\lambda)}\Big)\Big)\Big]\Big/K, \qquad \text{(A.4c)}$$

$$\bar{\phi} = \frac{(n+4)\overline{T}}{2\pi\overline{G}^0_{\theta z}\left(1-\bar{r}_i^{(n+4)}\right)} \quad \text{when } n \neq -4,$$
$$\bar{\phi} = -\frac{\overline{T}}{2\pi\overline{G}^0_{\theta z}\ln\bar{r}_i} \quad \text{when } n = -4, \qquad \text{(A.4d)}$$

where K is:

$$K = 2\pi\Big[\Big((n+1)(n+\lambda+2)(n-\lambda+2)v_{\theta z}\overline{E}^0_{zz}\mu + ((n+2)^2-\lambda^2)\overline{E}^0_{\theta\theta}(\overline{E}^0_{zz}+v_{zr}\mu)\Big)$$
$$\times\Big(\bar{r}_i^{(2n+2)} + \bar{r}_i^{(n+\lambda)} - \bar{r}_i^{(2n+\lambda+2)} - \bar{r}_i^n\Big) + 4(n+2)\overline{E}^0_{\theta\theta}\mu((n+\lambda+2)g_{-\lambda} - (n-\lambda+2)g_{\lambda})\bar{r}_i^{\left(\frac{3n+\lambda}{2}+1\right)}$$
$$+2(n+2)\overline{E}^0_{\theta\theta}\mu\Big((n-\lambda+2)g_{\lambda}(\bar{r}_i^{(2n+\lambda+2)} + \bar{r}_i^n) - (n+\lambda+2)g_{-\lambda}\Big(\bar{r}_i^{(2n+2)} + \bar{r}_i^{(n+\lambda)}\Big)\Big)\Big] \qquad \text{(A.5)}$$

and μ, g_{λ}, $g_{-\lambda}$ are given below:

$$\mu = \frac{4\overline{E}^0_{\theta\theta}(v_{\theta z}\overline{E}^0_{zz} - v_{zr}\overline{E}^0_{\theta\theta})}{(\overline{E}^0_{\theta\theta} - v^2_{\theta z}\overline{E}^0_{zz})((n+2)^2-\lambda^2)}, \qquad \text{(A.6a)}$$

$$g_{\lambda} = \nu_{zr} + \nu_{\theta z}\frac{(n+\lambda)\overline{E}^{0}_{zz}}{2\overline{E}^{0}_{\theta\theta}}, \tag{A.6b}$$

$$g_{-\lambda} = \nu_{zr} + \nu_{\theta z}\frac{(n-\lambda)\overline{E}^{0}_{zz}}{2\overline{E}^{0}_{\theta\theta}}. \tag{A.6c}$$

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